Execution of Multidisciplinary Design Optimization Approaches on Common Test Problems

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A class of synthetic problems for testing multidisciplinary design optimization (MDO) approaches is presented. These test problems are easy to reproduce because all functions are given as closed-form mathematical expressions. They are constructed in such a way that the optimal value of all variables and the objective is unity. The test problems involve three disciplines and allow the user to specify the number of design variables, state variables, coupling functions, design constraints, controlling design constraints, and the strength of coupling. Several MDO approaches were executed on two sample synthetic test problems. These approaches included single-level optimization approaches, collaborative optimization approaches, and concurrent subspace optimization approaches. Execution results are presented, and the robustness and efficiency of these approaches are evaluated for these sample problems.

Nomenclature

A_{coup}	= coupling factor between disciplines
A_{state}	= coupling factor for disciplinary state equations
$A1, A2, A3$	= disciplinary analyzers
$a, b, a1_{ij}, b1_{ij}, c1_{ij},$	= exponents
$d1_{ij}, a2_{ij}, b2_{ij},$	
$c2_{ij}, d2_{ij}, a3_{ij},$	
$b3_{ij}, c3_{ij}, d3_{ij}$	
$C, C1_i, C2_i, C3_i$	= constants
E	= Young's modulus for cantilever beam
$E1, E2, E3$	= disciplinary evaluators
$e1_{ij}, e2_{ij}, e3_{ij}$	= coefficients
f	= system objective function
$f1, f2, f3 \in \mathbb{R}^{nf}$	= disciplinary objective functions
$\hat{f}1, \hat{f}2, \hat{f}3$	= disciplinary objective variables
$g1, g2, g3 \in \mathbb{R}^{ng}$	= disciplinary design functions
$h1, h2, h3 \in \mathbb{R}^{nh}$	= disciplinary state functions
L	= cantilever beam length
nf	= number of objective functions in each discipline
ng	= number of design constraints in each discipline
ngc	= number of controlling design constraints in each discipline
nh	= number of state equations in each discipline
np	= number of functions in each discipline
nv	= number of variables in each discipline
nx	= number of design variables in each discipline
nxs	= number of system design variables
ny	= number of coupling functions from one discipline to another
$O1, O2, O3$	= disciplinary optimizers
P	= cantilever beam load
$p1, p2, p3 \in \mathbb{R}^{np}$	= disciplinary functions
$r1, r2, r3$	= disciplinary discrepancy functions
$\tilde{r}1$	= dummy disciplinary discrepancy variable
SA	= system analyzer
SO	= system optimizer
$v1, v2, v3 \in \mathbb{R}^{nv}$	= disciplinary variables
x_1, x_2	= variables
$x \in \mathbb{R}^{nxs}$	= system design variables
$\tilde{x} \in \mathbb{R}^{nxs}$	= dummy system design variables
$x1, x2, x3 \in \mathbb{R}^{nx}$	= disciplinary design variables
$y12, y13, y21, y23,$	= coupling functions
$y31, y32 \in \mathbb{R}^{ny}$	
$\hat{y}12, \hat{y}13, \hat{y}21, \hat{y}23,$	= coupling variables
$\hat{y}31, \hat{y}32 \in \mathbb{R}^{ny}$	
$\tilde{y}21, \tilde{y}31 \in \mathbb{R}^{ny}$	= dummy coupling variables
$z1, z2, z3 \in \mathbb{R}^{nh}$	= disciplinary state variables
δ	= allowable deflection for cantilever beam
λ	= Lagrange multiplier
σ	= allowable normal stress for cantilever beam
τ	= allowable shear stress for cantilever beam

Introduction

MULTIDISCIPLINARY design optimization (MDO) is a developing field of study that is concerned with how to optimally design and analyze systems composed of multiple disciplinary models that are coupled. Usually the design of such complex systems is performed by a team that is subdivided into groups associated with the disciplines. The disciplines in the system may correspond to fields of study (aerodynamics, structures, thermodynamics), or they may correspond to physical parts (wing, fuselage, engines). One goal of MDO is to allow the disciplinary groups to analyze and design in parallel with a certain degree of autonomy. Nevertheless, because the disciplines are coupled, their work must be coordinated at the system level so that an overall optimum design can be achieved for the system.

Several approaches for formulating and solving MDO problems have appeared in the last decade. Many of these approaches can be categorized into three groups. The first group consists of single-level optimization approaches.[1-4] In these approaches, optimization is performed only at the system level, and the role of the disciplines is limited to analysis and function evaluation. The second group consists of collaborative optimization approaches,[5,6] and the third group consists of concurrent subspace optimization approaches.[7-10] In these latter two groups, optimization is performed at the system level as well as within the disciplines. A major difference between concurrent subspace optimization and collaborative optimization is that in concurrent subspace optimization, each discipline attempts to satisfy its own constraints as well as approximations to the constraints of the other disciplines, whereas in collaborative optimization, each discipline satisfies its own constraints and tries to match

target values on coupling functions that are needed by other disciplines in the evaluation of their constraints.

The robustness and efficiency of the various approaches are not well understood. It is difficult to obtain good MDO test problems. Realistic test problems often involve cumbersome disciplinary software packages. This makes such test problems difficult to reproduce. Furthermore, results may be strongly influenced by interfacing details and the internal programming characteristics of the disciplinary analyses, and thus the inherent behavior of the MDO approach may be clouded.

Another deficiency in many MDO test problems is the lack of completeness. The number of disciplines or the numbers of different types of variables and functions in the disciplines may be small. Disciplinary models may be linear or otherwise trivial. The coupling between disciplines may be weak or incomplete. Results from the execution of MDO approaches on incomplete test problems may not be indicative of the results that can be expected on more challenging problems.

In this paper, a class of synthetic MDO test problems will be presented, and several MDO approaches will be executed on two common test problems. The synthetic test problems possess the following desirable characteristics.

1) They are easy to reproduce because all functions are given as closed-form mathematical expressions.

2) They are constructed in such a way that the optimal value of all variables and the objective is unity. The feasible region is convex, and this optimum is unique.

3) There are three disciplines, although the concepts presented can be extended to create synthetic test problems with more than three disciplines.

4) The completeness is controlled by the user by specifying the numbers of design variables, state variables, coupling functions, design constraints, and controlling design constraints.

5) Coupling between disciplines is complete, and the strength of the coupling is controlled by the user.

6) The functions are nonlinear and distinct; however, exact solution of the state and coupling equations is possible.

Sample Elements of an MDO Problem

We now define some of the elements of an MDO problem. We will refer to the three-discipline system shown in Fig. 1. We assume that all analytical models in the system are assigned to disciplines. The disciplinary models have inputs and outputs. All inputs and outputs shown in Fig. 1 are vectors and are defined in the Nomenclature. We assume that the optimization problem to be solved is defined as follows.

1) Disciplinary state equations must be satisfied: $h1 = 0$, $h2 = 0$, and $h3 = 0$. These equations may be equations of equilibrium, compatibility, constitution, conservation, etc., based on the physical model for the discipline. Note that the number of disciplinary state variables is assumed to be equal to the number of disciplinary state functions. This implies that the state equations for each discipline can be used to solve for the values of the state variables (all other variables being held constant).

2) Disciplinary design constraints must be satisfied: $g1 \leq 0$, $g2 \leq 0$, and $g3 \leq 0$. These constraints distinguish acceptable/feasible designs from unacceptable/failed designs.

3) The system objective function must be minimized: minimize $f = f(f1, f2, f3)$. Note that the system objective is a scalar function of the disciplinary objective functions.

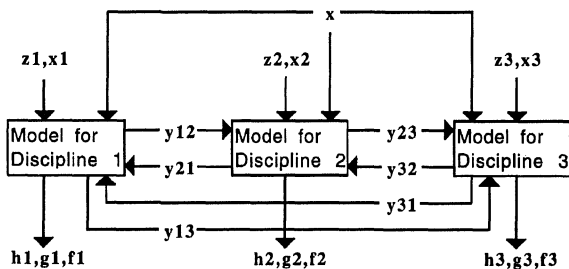4) Coupling equations must be satisfied: $y12 = \hat{y}12$, $y13 = \hat{y}13$, $y21 = \hat{y}21$, $y23 = \hat{y}23$, $y31 = \hat{y}31$, and $y32 = \hat{y}32$. There is a coupling variable associated with each coupling function. These coupling variables make it possible to evaluate disciplinary models in parallel in a coupled system. The disciplines use assumed values of coupling variables as input to the disciplinary models and evaluate their output, which includes the coupling functions.

Synthetic Test Problems

Figure 2 plots curves for the equation $x_1^a x_2^b - 1 = 0$ for various negative values of the exponents a and b. We propose synthetic design constraints of the form $g = x_1^a x_2^b - C \leq 0$ with $a < 0$, $b < 0$, and $C \geq 1$. The feasible region for such constraints is convex, and the point $x_1 = x_2 = 1$ is feasible. We take $C = 1$ for the constraints that control at the optimum and we take $C > 1$ for the constraints that do not control at the optimum.

Consider a structural optimization problem consisting of a cantilever beam of length L with a transverse load P at the free end. The cross section is rectangular and the design variables x_1 and x_2 are, respectively, the section width and depth. If the allowable values for deflection, normal stress, and shear stress are δ, σ, and τ, respectively, then the constraints are as follows.

Deflection:

$$g_1 = (4PL^3/E)x_1^{-1}x_2^{-3} - \delta \leq 0$$

Normal stress:

$$g_2 = 6PLx_1^{-1}x_2^{-2} - \sigma \leq 0 \tag{1}$$

Shear stress:

$$g_3 = (3P/2)x_1^{-1}x_2^{-1} - \tau \leq 0$$

Thus, we see that the proposed synthetic constraints resemble the form of constraints that may be encountered in structural optimization.

We now give formulas for the design functions in our synthetic test problems. First, we define the vector of variables $v1$ for discipline 1. The size of this vector is $nv = nxs + nx + nh + 2ny$. The vector $v1$ contains the values of the input variables for discipline 1 in the following order: $x, x1, z1, \hat{y}21, \hat{y}31$. In the following formula for the design functions of discipline 1, the subscripts i or j to the right of a vector indicate the ith or jth element of the vector:

$$g1_i = \prod_{j=1}^{nv}(v1_j)^{a1_{ij}} - C1_i \tag{2}$$

We take $C1_i = 1$ for $i \leq ngc$, and we take $C1_i = 1.2$ for $ngc < i \leq ng$. We randomly generate values in the interval $(-5, 0)$ for each of the exponents $a1_{ij}$. Similar formulas can be constructed for the design functions for disciplines 2 and 3.

We formally define the state functions in similar fashion. For discipline 1,

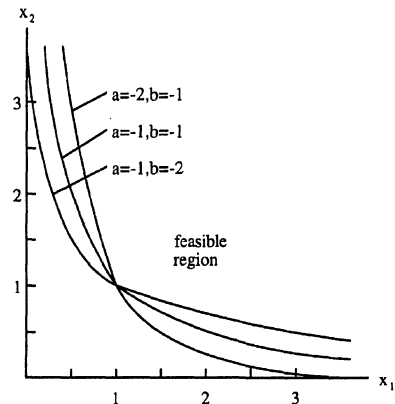$$h1_i = \prod_{j=1}^{nv}(v1_j)^{b1_{ij}} - 1 \tag{3}$$

Fig. 1   Coupled system with three disciplines.

Fig. 2   Curves of the equation $x_1^a x_2^b - 1 = 0$.

Note that the state equations $h1 = 0$ will be satisfied when all design, state, and coupling variables are equal to 1 regardless of whether the exponents are positive or negative. We have chosen to use positive exponents for state functions. However, if the exponents are too large, the magnitude of the state functions could blow up. This can be prevented by linking the size of the interval from which exponents are randomly generated to the number of variables in the product. For the state function $h1_i$, the exponents for the variables are determined as follows.

Variable: Exponent $b1_{ij}$:

$x_j, x1, \hat{y}21_j, \hat{y}31_j$ random from interval $\left(0, \dfrac{1}{nxs + nx + 2ny}\right)$

$z1_j\ j \neq i$ random from interval $\left(0, \dfrac{A_{state}}{nh}\right)$

$z1_j\ j = i$ 1

The term A_{state} is a user-specified factor that controls the strength of coupling in the state equations. If $A_{state} = 0$, the ith state function is a function of the ith state variable only and not of the other state variables. If $A_{state} > 0$, then each state function is coupled to all of the state variables.

We formally define the coupling functions in similar fashion. For the coupling functions evaluated in discipline 1,

$$y12_i = \prod_{j=1}^{nv}(v1_j)^{c1_{ij}} \qquad y13_i = \prod_{j=1}^{nv}(v1_j)^{d1_{ij}} \qquad (4)$$

Variable: Exponent $c1_{ij}$ and $d1_{ij}$:

$x_j, x1_j, z1_j$ random from interval $\left(0, \dfrac{1}{nxs + nx + nh}\right)$

$\hat{y}21_j, \hat{y}31_j$ random from interval $\left(0, \dfrac{A_{coup}}{2ny}\right)$

$$e1_j = -3\lambda \begin{cases} \displaystyle\sum_{i=1}^{ngc}a1_{ij} + \sum_{i=1}^{nh}b1_{ij} + \sum_{i=1}^{ny}c1_{ij} + \sum_{i=1}^{ny}d1_{ij} & \text{for} \quad j \leq nxs + nx + nh \\[2ex] \displaystyle\sum_{i=1}^{ngc}a1_{ij} + \sum_{i=1}^{nh}b1_{ij} + \sum_{i=1}^{ny}c1_{ij} + \sum_{i=1}^{ny}d1_{ij} - 1 & \text{for} \quad j > nxs + nx + nh \end{cases} \qquad (9)$$

The term A_{coup} is a user-specified factor that controls the strength of coupling between disciplines. Note that if $A_{coup} = 0$, the coupling functions in one discipline are not functions of the coupling variables from the other disciplines.

$$f = \frac{1}{6}\sum_{j=1}^{nv}(e1_j + e2_j + e3_j) = \frac{\lambda}{2}\left\{6ny - \sum_{j=1}^{nv}\left[\sum_{i=1}^{ngc}(a1_{ij} + a2_{ij} + a3_{ij}) + \sum_{i=1}^{nh}(b1_{ij} + b2_{ij} + b3_{ij}) \atop + \sum_{i=1}^{ny}(c1_{ij} + c2_{ij} + c3_{ij} + d1_{ij} + d2_{ij} + d3_{ij})\right]\right\} = 1 \qquad (10)$$

We assume that there is a single objective function for each discipline, and we define the system objective function to be the average of the disciplinary objective functions:

$$f = \tfrac{1}{3}(f1 + f2 + f3) \qquad (5)$$

We employ a quadratic form for the disciplinary objective functions. Thus, for discipline 1,

$$f1 = \frac{1}{2}\sum_{j=1}^{nv}e1_j(v1_j)^2 \qquad (6)$$

The coefficients $e1_j$ are determined such that the point where all design, state, and coupling variables are equal to unity is the optimum. Therefore, the Kuhn–Tucker conditions require that at the optimum

$$\frac{1}{3}(\nabla f1 + \nabla f2 + \nabla f3)$$

$$+ \lambda \begin{bmatrix} \displaystyle\sum_{i=1}^{ngc}\nabla g1_i + \sum_{i=1}^{ngc}\nabla g2_i + \sum_{i=1}^{ngc}\nabla g3_i + \sum_{i=1}^{nh}\nabla h1_i \\[1.5ex] + \displaystyle\sum_{i=1}^{nh}\nabla h2_i + \sum_{i=1}^{nh}\nabla h3_i + \sum_{i=1}^{ny}\nabla(y12 - \hat{y}12)_i \\[1.5ex] + \displaystyle\sum_{i=1}^{ny}\nabla(y13 - \hat{y}13)_i + \sum_{i=1}^{ny}\nabla(y21 - \hat{y}21)_i \\[1.5ex] + \displaystyle\sum_{i=1}^{ny}\nabla(y23 - \hat{y}23)_i + \sum_{i=1}^{ny}\nabla(y31 - \hat{y}31)_i \\[1.5ex] + \displaystyle\sum_{i=1}^{ny}\nabla(y32 - \hat{y}32)_i \end{bmatrix} = 0$$

$$(7)$$

Note that we have assumed that the values of all nontrivial Lagrange multipliers are equal to the positive scalar value λ, which is yet to be determined. In the preceding equation, $\nabla f1$ is the gradient of $f1$ with respect to all design, state, and coupling variables in the system, and $\nabla g1_i$ is the gradient of the ith element of $g1$ with respect to all design, state, and coupling variables in the system. The preceding optimality equation is satisfied if optimality equations are satisfied for each discipline. For discipline 1 the optimality equation is

$$\frac{1}{3}\nabla f1 + \lambda\left[\sum_{i=1}^{ngc}\nabla g1_i + \sum_{i=1}^{nh}\nabla h1_i + \sum_{i=1}^{ny}\nabla y12_i \right.$$
$$\left. + \sum_{i=1}^{ny}\nabla y13_i - \sum_{i=1}^{ny}\nabla\hat{y}21_i - \sum_{i=1}^{ny}\nabla\hat{y}31_i\right] = 0 \qquad (8)$$

In the preceding equation, the gradients are with respect to $v1$. Evaluating the gradients of Eqs. (2–6) at $v1 = 1$, the disciplinary optimality equation leads to formulas for the coefficients $e1_j$ in terms of exponents $a1_{ij}$, $b1_{ij}$, $c1_{ij}$, and $d1_{ij}$:

Similar formulas are developed for disciplines 2 and 3. The requirement that the system objective have a value of unity at the optimum leads to the following equation for determining the Lagrange multiplier:

The value of the Lagrange multiplier computed from the preceding formula should be checked for positivity.

Implementation

Implementation of the synthetic test problems involves an initialization subroutine that reads from the user the following numbers and factors: $nxs, nx, nh, ny, ng, ngc, A_{state}$, and A_{coup}. The initialization subroutine then calls a random number generation subroutine to get the exponents for the three disciplines $a1_{ij}, b1_{ij}, c1_{ij}, d1_{ij}, a2_{ij}, b2_{ij}, c2_{ij}, d2_{ij}, a3_{ij}, b3_{ij}, c3_{ij}$, and $d3_{ij}$ as described in the previous section. Finally, the initialization subroutine computes the Lagrange multiplier from Eq. (10) and the coefficients $e1_j$, $e2_j$, and $e3_j$ for the objective functions according to Eq. (9). These coefficients and exponents are stored.

Implementation also includes an evaluation subroutine for each of the three disciplines. We define the vector of functions $p1$ for discipline 1. The size of this vector is $np = ng + nh + 2ny + 1$. The vector $p1$ contains the values of the output functions for discipline 1 in the following order: $g1, h1, y12, y13, f1$. The evaluation subroutine for discipline 1 receives the argument $v1$ and returns the argument $p1$. The functions are computed according to Eqs. (2–4) and (6).

Explicit gradient evaluation subroutines can also be written for each discipline. The gradient evaluation subroutine for discipline 1 receives the argument $v1$ and returns the arguments $p1$ and $\nabla p1$ where $\nabla p1$ is the matrix of derivatives of the elements of $p1$ with respect to the elements of $v1$. Equations (2–4) and (6) are easily differentiated in closed form:

$$\nabla g1_{ij} = \frac{a1_{ij}(g1_i + C1_i)}{v1_j} \qquad \nabla h1_{ij} = \frac{b1_{ij}(h1_i + 1)}{v1_j}$$

$$\nabla y12_{ij} = \frac{c1_{ij}y12_i}{v1_j} \qquad \nabla y13_{ij} = \frac{d1_{ij}y13_i}{v1_j} \qquad (11)$$

$$\nabla f1_j = e1_j v1_j$$

Single-Level Optimization Approaches

Six single-level optimization approaches were executed on synthetic MDO problems in the study. The first approach has been called the all-at-once approach[11] or the simultaneous analysis and design approach.[12] Using the notation introduced in Ref. 13, the first approach is

$$SO[E1 \parallel E2 \parallel E3]$$

The disciplinary evaluator calls the evaluation subroutine for the discipline. Thus, the disciplinary evaluator for discipline 1 performs the following task.

Given:

$$x, x1, z1, \hat{y}21, \hat{y}31$$

Return:

$$g1, h1, y12, y13, f1$$

The symbol $\parallel$ indicates that the disciplinary evaluators are executed in parallel. A sequential quadratic programming (SQP) algorithm[14] was used as the system optimizer for all six single-level optimization approaches in the study. The symbol $[\cdots]$ indicates nested execution meaning that the evaluation subroutines are executed at each iteration of the system optimizer. The optimization problem solved by the system optimizer in the all-at-once approach is as follows.

Find:

$$x, x1, x2, x3, z1, z2, z3, \hat{y}12, \hat{y}13, \hat{y}21, \hat{y}23, \hat{y}31, \hat{y}32$$

Minimize:

$$\frac{f1 + f2 + f3}{3}$$

Satisfy:

$g1 \le 0$	$g2 \le 0$	$g3 \le 0$	design constraints
$h1 = 0$	$h2 = 0$	$h3 = 0$	state equations
$y12 = \hat{y}12$	$y13 = \hat{y}13$	$y21 = \hat{y}21$	coupling equations
$y23 = \hat{y}23$	$y31 = \hat{y}31$	$y32 = \hat{y}32$	coupling equations

The second single-level optimization approach in the study has been called the individual discipline feasible approach[11] or the nested analysis and design approach.[13] The notation for this approach is

$$SO[A1[E1] \parallel A2[E2] \parallel A3[E3]]$$

The system optimizer solves the same optimization problem as in the all-at-once approach except that the state variables and state equations are eliminated. The role of $A1$, $A2$, and $A3$ is to solve the state equations for the state variables in each discipline. Thus, the disciplinary analyzer for discipline 1 solves the following problem.

Given:

$$x, x1, \hat{y}21, \hat{y}31$$

Find:

$$z1$$

Satisfy:

$$h1 = 0$$

Return:

$$g1, y12, y13, f1$$

In this study, the disciplinary analyzers employed a Newton iteration algorithm to solve the state equations. The disciplinary analyzers were also modified to explicitly calculate gradients. It might be pointed out that it is possible to construct a direct rather than iterative disciplinary analyzer by taking the log of Eq. (3) to get a set of linear equations that can be solved directly for the logs of the state variables. Because direct disciplinary analyzers do not call disciplinary evaluators, the notation for such an approach would be $SO[A1 \parallel A2 \parallel A3]$.

The third single-level optimization approach in the study has been called the multidiscipline feasible approach.[11] The notation for this approach is

$$SO[SA[A1[E1] \parallel A2[E2] \parallel A3[E3]]]$$

The system optimizer solves the same optimization problem as in the all-at-once approach except that the state variables, coupling variables, state equations, and coupling equations are eliminated. The system analyzer solves the coupling equations for the values of the coupling variables. Specifically, it solves the following problem.

Given:

$$x, x1, x2, x3$$

Find:

$$\hat{y}12, \hat{y}13, \hat{y}21, \hat{y}23, \hat{y}31, \hat{y}32$$

Satisfy:

$y12 = \hat{y}12$	$y13 = \hat{y}13$	$y21 = \hat{y}21$
$y23 = \hat{y}23$	$y31 = \hat{y}31$	$y32 = \hat{y}32$

Return:

$$g1, g2, g3, f1, f2, f3$$

Two iterative system analyzers are possible. The first is a fixed-point iteration algorithm in which the computed values of the coupling functions at any iteration are used as the assumed values for the coupling variables in the next iteration. The second is a gradient-based Newton iteration algorithm that converges in slightly fewer iterations than fixed-point iteration. The Newton iteration algorithm was used in the study. The system analyzer was modified to explicitly calculate gradients based on the global sensitivity equations.[15] It is possible to construct a direct rather than iterative system analyzer by taking the log of Eq. (4) to get a set of linear equations that can be solved directly for the logs of the coupling variables. Because the direct system analyzer does not call disciplinary analyzers or evaluators, the notation for such an approach would be $SO[SA]$.

The fourth single-level optimization approach in the study is also a multidiscipline feasible approach like the previous approach. The difference between the approaches is evident from the notation for the fourth approach:

$$SO[SA[A1[E1] \rightarrow A2[E2] \rightarrow A3[E3]]]$$

The symbol $\rightarrow$ indicates serial execution of the disciplinary analyzers. Because $A1$ is executed before $A2$ and $A3$, there is no need for coupling variables $\hat{y}12$ and $\hat{y}13$ because the coupling functions $y12$ and $y13$ could be sent directly from $A1$ to $A2$ and $A3$. Similarly, there is no need for coupling variables $\hat{y}23$ because $A2$ is

executed before $A3$. Thus, these coupling variables and their corresponding coupling equations are deleted from the problem solved by the system analyzer in this approach.

The fifth single-level optimization approach in the study is also a multidiscipline feasible approach like the previous two approaches. However, the notation for the fifth approach is different:

$$SA[A1[E1] \parallel A2[E2] \parallel A3[E3]]$$

$$\leftrightarrow SO[A1[E1] \parallel A2[E2] \parallel A3[E3]]$$

The symbol $\leftrightarrow$ indicates alternating execution of the system analyzer and the system optimizer. This means that the design variables are held constant while the system analyzer solves for the coupling variables, and the coupling variables are held constant while the system optimizer solves for the design variables. This process is repeated for a specified number of cycles.

The sixth single-level optimization approach in the study is also a multidiscipline feasible approach and has the same notation as the fifth approach. It is identical to the fifth approach with one exception: the coupling variables are not held constant during system optimization, but rather they are approximated as linear functions of the design variables based on values of derivatives provided by the system analyzer.

Collaborative Optimization Approaches

The collaborative optimization approaches that were considered in the study are all described by the following notation:

$$SO[O1[E1] \parallel O2[E2] \parallel O3[E3]]$$

This implies that in collaborative optimization, disciplines are permitted to design as well as analyze. The problem solved by the system optimizer in collaborative optimization is as follows.
Find:

$$x, \hat{y}12, \hat{y}13, \hat{y}21, \hat{y}23, \hat{y}31, \hat{y}32, \hat{f}1, \hat{f}2, \hat{f}3$$

Minimize:

$$\frac{\hat{f}1 + \hat{f}2 + \hat{f}3}{3}$$

Satisfy:

$$r1 \leq 0 \qquad r2 \leq 0 \qquad r3 \leq 0$$

The role of system optimization is the determination of system design variables, coupling variables between disciplines, and the disciplinary objective variables, $\hat{f}1$, $\hat{f}2$, $\hat{f}3$, which serve as target values on the objective functions for each of the disciplines. The system receives only a single scalar function back from each discipline. We will call these functions $(r1, r2, r3)$ discrepancy functions. They are a measure of how successful the disciplines were in solving their optimization problems. A value of zero indicates complete success, whereas a positive value indicates that success was incomplete.

Three formulations for the disciplinary optimization problem were considered in the study. In formulation 1, the optimizer for discipline 1 solves the following problem.
Given:

$$x, \hat{y}21, \hat{y}31, \hat{y}12, \hat{y}13, \hat{f}1$$

Find:

$$x1, z1, \tilde{x}, \tilde{y}21, \tilde{y}31$$

Minimize:

$$r1 = \sqrt{\begin{array}{l} \sum_{i=1}^{ny}(y12_i - \hat{y}12_i)^2 + \sum_{i=1}^{ny}(y13_i - \hat{y}13_i)^2 \\ + (f1 - \hat{f}1)^2 + \sum_{i=1}^{nxs}(\tilde{x}_i - \hat{x}_i)^2 + \sum_{i=1}^{ny}(\tilde{y}21_i - \hat{y}21_i)^2 \\ + \sum_{i=1}^{ny}(\tilde{y}31_i - \hat{y}31_i)^2 \end{array}}$$

Satisfy:

$$g1 \leq 0 \qquad h1 \leq 0$$

Return:

$$r1$$

In this formulation, the evaluator for discipline 1 performs the following task.
Given:

$$\tilde{x}, x1, z1, \tilde{y}21, \tilde{y}31$$

Return:

$$g1, h1, y12, y13, f1$$

If one carefully studies the preceding, one recognizes that in formulation 1 discipline 1 satisfies its own design constraints and state equations and does its best to match the dummy input variables to the evaluator $(\tilde{x}, \tilde{y}21, \tilde{y}31)$ with system targets $(x, \hat{y}21, \hat{y}31)$ and to match output functions from the evaluator $(y12, y13, f1)$ with system targets $(\hat{y}12, \hat{y}13, \hat{f}1)$. An l_2 (Euclidean) metric is used to quantify the success in matching the target values.

In formulation 2 of collaborative optimization, the optimizer for discipline 1 solves the following problem.
Given:

$$x, \hat{y}21, \hat{y}31, \hat{y}12, \hat{y}13, \hat{f}1$$

Find:

$$x1, z1, \tilde{x}, \tilde{y}21, \tilde{y}31, \tilde{r}1$$

Minimize:

$$r1 = \tilde{r}1$$

Satisfy:

$$g1 \leq 0 \qquad h1 = 0$$

$$|y12_i - \hat{y}12_i| \leq \tilde{r}1 \qquad |y13_i - \hat{y}13_i| \leq \tilde{r}1 \qquad |f1 = \hat{f}1| \leq \tilde{r}1$$

$$|\tilde{x}_i - \hat{x}_i| \leq \tilde{r}1 \qquad |\tilde{y}21_i - \hat{y}21_i| \leq \tilde{r}1 \qquad |\tilde{y}31_i - \hat{y}31_i| \leq \tilde{r}1$$

Return:

$$r1$$

The evaluator for discipline 1 performs the same task as in formulation 1. Note that in formulation 2 the l_∞ (max) metric of the mismatches with the system targets is minimized rather than the l_2 metric. This is accomplished by minimizing the dummy variable $\tilde{r}1$, which is constrained to be greater than all mismatches. This is the only difference between formulations 1 and 2.

In formulation 3 of collaborative optimization, the optimizer for discipline 1 solves the following problem.
Given:

$$x, \hat{y}21, \hat{y}31, \hat{y}12, \hat{y}13, \hat{f}1$$

Find:

$$x1, z1, \tilde{x}, \tilde{r}1$$

Minimize:

$$r1 = \tilde{r}1$$

Satisfy:

$$g1_i \leq \tilde{r}1 \qquad |h1_i| \leq \tilde{r}1$$

$$|y12_i - \hat{y}12_i| \leq \tilde{r}1 \qquad |y13_i - \hat{y}13_i| \leq \tilde{r}1 \qquad |f1 = \hat{f}1| \leq \tilde{r}1$$

Return:

$$r1$$

In this formulation, the evaluator for discipline 1 performs the following task.
Given:

$$x, x1, z1, \hat{y}21, \hat{y}31$$

Return:

$$g1, h1, y12, y13, f1$$

Note that in this formulation the input variables to the evaluator are the system targets so that there are no mismatches on the inputs. The l_∞ metric of the mismatches of the outputs and of the violation in design constraints and state equations is minimized.

Summarizing these collaborative optimization formulations, the disciplinary optimizers seek disciplinary design and state variables that satisfy three goals: 1) match target inputs to the disciplinary model, 2) match target outputs from the disciplinary model, and 3) satisfy disciplinary design constraints and state equations. In general, it is possible to satisfy completely only one of the three goals for all possible target values sent down from the system. In formulations 1 and 2, the third goal is satisfied completely, whereas a metric of the discrepancy in the first two goals is minimized. In formulation 3, the first goal is satisfied completely, whereas a metric of the discrepancy in the last two goals is minimized. In formulation 1, the l_2 metric is employed, whereas in formulations 2 and 3, the l_∞ metric is employed.

In the study, the SQP algorithm was used to solve the disciplinary optimization problems for all three formulations. However, the SQP algorithm and a cutting plane algorithm[6] were used as system optimizers for each of the three formulations. The latter algorithm was used because the discrepancy functions received from the disciplines may be nonsmooth or even discontinuous functions in collaborative optimization.

The beauty of collaborative optimization is that each discipline is in control of its own design/state variables/functions and does not meddle directly with the design/state variables/functions of the other disciplines. The system level does not meddle directly with the design/state variables/functions of the disciplines, but rather it focuses on the system design variables, the coupling variables between disciplines, and the allocation of objective responsibility among the disciplines. These qualities of collaborative optimization make it amenable to existing organizational structures in industry. Furthermore, these qualities give the disciplines the autonomy to search over topologies and concepts, modify their models, and even reformulate their variables and functions without disrupting the MDO process.

Concurrent Subspace Optimization

Concurrent subspace optimization is formulated for problems without system design variables x. As alluded to earlier, this does not limit generality because the system design variables could be included in the design variables for one of the disciplines, and their values could be passed as coupling functions to the other disciplines.

The concurrent subspace optimization approaches that were considered in the study are described by the following notation:

$$SA[A1[E1] \parallel A2[E2] \parallel A3[E3]]$$

$$\leftrightarrow O1[A1[E1]] \parallel O2[A2[E2]] \parallel O3[A3[E3]]$$

It is implied that nested execution $[\cdots]$ has precedence over parallel execution $\parallel$, which has precedence over alternating execution $\leftrightarrow$. The system analyzer solves the coupling equations for the coupling variables with the design variables held fixed. Two formulations for the disciplinary optimization problems were considered in the study. For both formulations, the SQP algorithm was used to solve the disciplinary optimization problems. In the first formulation, the optimizer for discipline 1 solves the following problem.

Find:
$$x1$$
Minimize:
$$f1$$
Satisfy:
$$g1 \le 0$$

In this formulation, linear approximations for coupling variables $\hat{y}21$ and $\hat{y}31$ are used in the evaluation of $f1$ and $g1$. These approximations are based on the values of derivatives with respect to $x1$ received from the system analyzer. Note that in this formulation each discipline determines its design variables without regard for the objectives and constraints in the other disciplines. We will refer to this formulation as the selfish linear formulation. It may reach the system optimum if the system is uncoupled or weakly coupled.

In the second formulation, the objectives and constraints from all disciplines are considered by each discipline. Thus, the optimizer for discipline 1 solves the following problem.

Find:
$$x1$$
Minimize:
$$\frac{f1 + f2 + f3}{3}$$
Satisfy:
$$g1 \le 0 \qquad g2 \le 0 \qquad g3 \le 0$$

As was the case in the selfish linear formulation, linear approximations for coupling variables $\hat{y}21$ and $\hat{y}31$ are used in the evaluation of $f1$ and $g1$ in the preceding formulation. Also, the optimizer for discipline 1 calls the analyzer for discipline 1 only and not the analyzers for disciplines 2 and 3. Therefore, linear approximations for the functions $f2$, $f3$, $g2$, and $g3$ are employed based on the values of derivatives of these functions with respect to $x1$ received from the system analyzer. We will refer to this formulation as the selfless linear formulation.

It is clear that the selfless linear formulation will not always converge to the system optimum. This is because it may be impossible for some disciplines to satisfy the approximate constraints of all disciplines. Surely this is the case if the disciplines are uncoupled. Researchers have relaxed the requirement that each discipline must completely satisfy the approximate constraints of all disciplines by introducing responsibility fractions, which are determined by solving a system optimization problem known as the coordination problem.[7-10] However, these approaches have not been implemented in the study.

Test Results

Two test problems were considered in the study. They are described by the following parameter values. Specific values for the hundreds of exponents randomly generated for these two problems can be obtained directly from the authors. Furthermore, these test problems are in the process of being placed on the internet under the test suite being sponsored by the NASA Langley MDO Branch.[16]

Problem 1

$nxs = 0$
$nx = 8$
$nh = 0$
$ny = 2$
$ng = 10$
$ngc = 6$
$A_{state} = 1.0$
$A_{coup} = 1.0$

Problem 2

$nxs = 2$
$nx = 8$
$nh = 10$
$ny = 2$
$ng = 10$
$ngc = 6$
$A_{state} = 1.0$
$A_{coup} = 1.0$

All approaches were executed on problem 1, but the concurrent subspace optimization approaches were not executed on problem 2 because it involves system design variables.

Maximum and minimum values of 5.0 and 0.5, respectively, were used for all variables. Starting values of 5.0 were used for all variables, although starting values of 0.5 and 2.0 were also tried with little change in results. Recall that the optimum values for all variables and for the objective are unity.

An evaluation counter was incremented in each call to the evaluation subroutine or the gradient evaluation subroutine for discipline 1. We assume that the number of calls to the evaluation subroutines

<div align="center">Table 1 Evaluation counts for convergence</div>

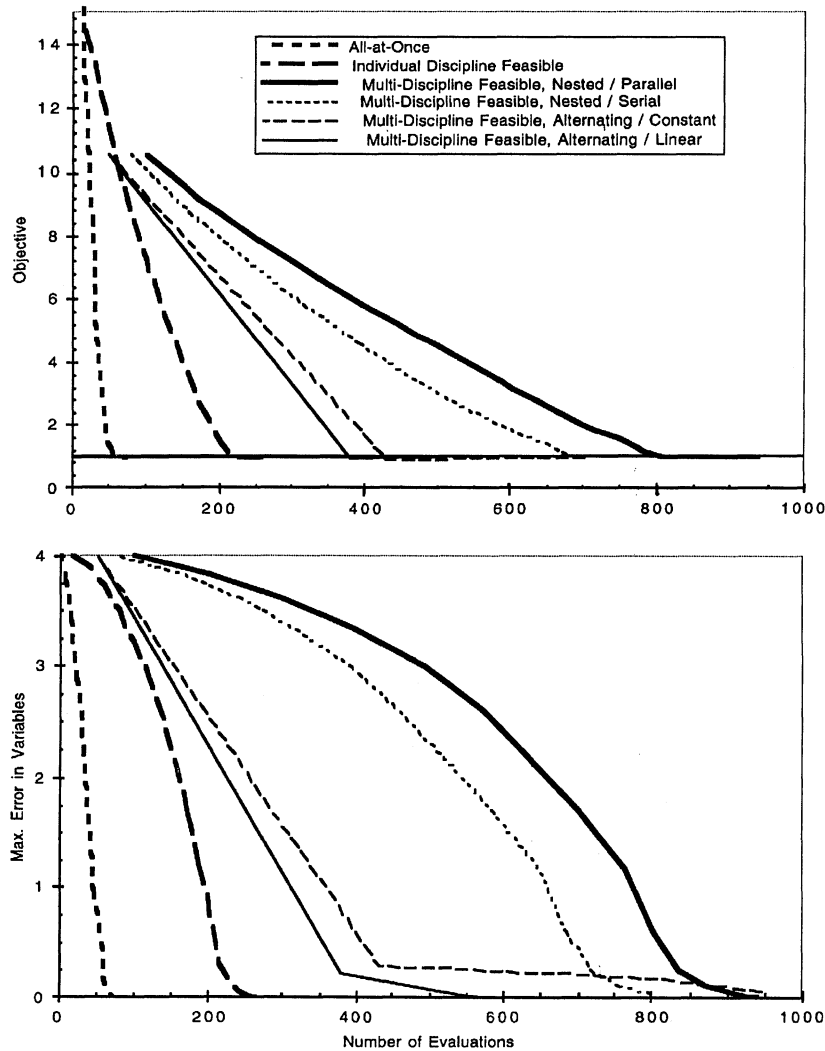| Approach	Problem 1	Problem 2
Single-level, all-at-once $SO[E1 \parallel E2 \parallel E3]$	48/60	56/65
Single-level, individual discipline feasible $SO[A1[E1] \parallel A2[E2] \parallel A3[E3]]$	48/60	216/240
Single-level, multidiscipline feasible, nested parallel $SO[SA[A1[E1] \parallel A2[E2] \parallel A3[E3]]]$	Failed	800/908
Single-level, multidiscipline feasible, nested serial $SO[SA[A1[E1] \rightarrow A2[E2] \rightarrow A3[E3]]]$	244/268	684/792
Single-level, multidiscipline feasible, alternating constant $SA[A1[E1] \parallel A2[E2] \parallel A3[E3]] \leftrightarrow SO[A1[E1] \parallel A2[E2] \parallel A3[E3]]$	Failed	430/946
Single-level, multidiscipline feasible, alternating linear $SA[A1[E1] \parallel A2[E2] \parallel A3[E3]] \leftrightarrow SO[A1[E1] \parallel A2[E2] \parallel A3[E3]]$	Failed	378/548
Collaborative, formulation 1, SQP algorithm $SO[O1[E1] \parallel O2[E2] \parallel O3[E3]]$	5,501/1,270	8,240/2,364
Collaborative, formulation 2, SQP algorithm $SO[O1[E1] \parallel O2[E2] \parallel O3[E3]]$	12,580/12,580	9,534/9,909
Collaborative, formulation 3, SQP algorithm $SO[O1[E1] \parallel O2[E2] \parallel O3[E3]]$	7,744/3,533	9,064/7,709
Collaborative, formulation 1, cutting plane algorithm $SO[O1[E1] \parallel O2[E2] \parallel O3[E3]]$	467/1,018	820/2,354
Collaborative, formulation 2, cutting plane algorithm $SO[O1[E1] \parallel O2[E2] \parallel O3[E3]]$	802/1,960	1,085/3,745
Collaborative, formulation 3, cutting plane algorithm $SO[O1[E1] \parallel O2[E2] \parallel O3[E3]]$	1,443/3,062	2,399/5,255
Concurrent subspace, selfish linear $SA[A1[E1] \parallel A2[E2] \parallel A3[E3]] \leftrightarrow O1[A1[E1]] \parallel O2[A2[E2]] \parallel O3[A3[E3]]$	Failed	Does not apply
Concurrent subspace, selfless linear $SA[A1[E1] \parallel A2[E2] \parallel A3[E3]] \leftrightarrow O1[A1[E1]] \parallel O2[A2[E2]] \parallel O3[A3[E3]]$	Failed	Does not apply

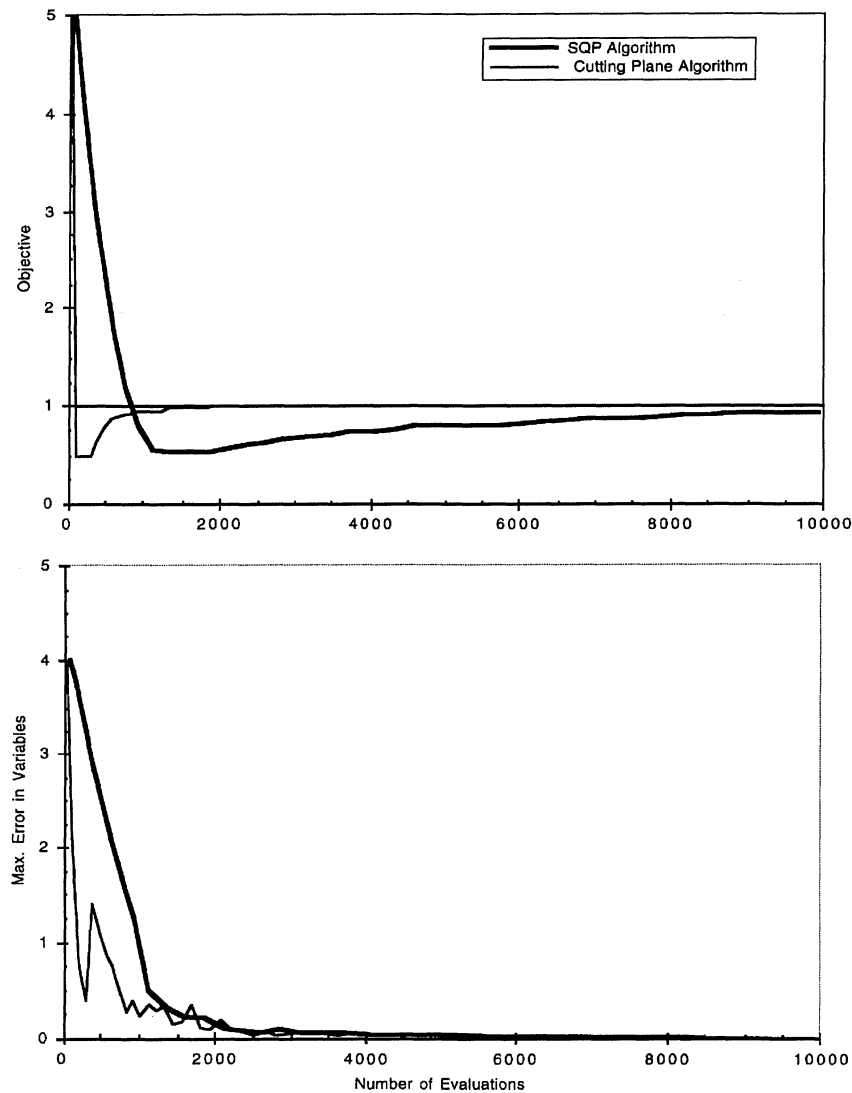Fig. 3   Performance of single-level optimization approaches on problem 2.

Fig. 4 Performance of collaborative optimization formulation 1 on problem 2.

for disciplines 2 and 3 would be roughly the same. Explicit gradient evaluation was used for all approaches (no finite difference gradients). Recall that for those approaches involving disciplinary analyzers a Newton method was used to solve the state equations, and for those approaches involving a system analyzer, a Newton method was used to solve the coupling equations. Newton methods were used rather than exact solvers, which would unfairly exploit the mathematical form of the synthetic problems and thus bias the comparison. Table 1 shows two numbers for each approach for each test problem. The left number is the value of the evaluation counter when objective reached and stayed within 1 ± 0.1, and the right number is the value of the evaluation counter when all variables reached and stayed within 1 ± 0.1.

A main break point was defined for each approach. The main break point for the first four single-level optimization approaches and for the collaborative optimization approaches was in the main iteration loop of the system optimizer. For the concurrent subspace optimization approaches and the last two single-level optimization approaches, the main break point was taken at the end of the system analyzer. At the main break point for each approach, the following three numbers were output: 1) the evaluation counter, 2) the value of the system objective, and 3) the maximum error in variables from optimal value of unity. The system objective and the maximum error in variables are plotted vs evaluation counter in Figs. 3 and 4 for problem 2.

Observations and Conclusions

Approach robustness seems to depend on the strength of coupling. Note in Table 1 which approaches failed at $A_{coup} = A_{state} = 1$. When

the coupling was increased to $A_{coup} = A_{state} = 2$, all approaches failed except the first two single-level optimization approaches and the collaborative optimization approaches. Perhaps these robust approaches could solve problems with even higher levels of coupling. All approaches succeeded when the coupling was decreased to $A_{coup} = A_{state} = \frac{1}{2}$ with the exception of the concurrent subspace selfless linear approach. Failure of the single-level multidiscipline feasible approaches occurred in the Newton system analyzer when negative values of the coupling variables were encountered during the Newton iterations. The same type of failure was observed in the concurrent subspace selfish linear approach. Thus, increased coupling causes difficulty in approaches involving system analyzers. Failure of the concurrent subspace selfless linear approach, even at low levels of coupling, occurred in the disciplinary optimizers, which were unable to find feasible designs. Clearly, this approach is flawed.

When started from the optimum, all of the single-level and concurrent subspace optimization approaches recognize optimality after gradients are calculated and immediately stop. However, when the collaborative optimization approaches are started from the optimum, they send the disciplinary objective variables to their lower bounds and work back to the optimum from the infeasible side just as if they were started from any other point. At the optimum, the discrepancy functions and their gradients with respect to the variables in the system optimization problem are zero. Since the gradient of the objective function is not zero at the optimum, the Kuhn–Tucker conditions as typically formulated are not satisfied. It should be pointed out that infinitesimally on the infeasible side of the optimum the gradients of the discrepancy functions are nonzero, and the Kuhn–Tucker conditions can be satisfied. Theoretical work is

needed to establish proper optimality conditions for terminating the collaborative optimization approaches.

Defining approach efficiency in terms of number of model evaluations, the single-level optimization approaches are clearly more efficient than the collaborative optimization approaches. The ranking of the single-level optimization approaches can be deduced from Fig. 3, where we see that the most efficient approach (all-at-once) requires about an order of magnitude fewer evaluations than the least efficient approach (multidiscipline feasible, nested parallel). The efficiency of the collaborative optimization approaches is improved on these test problems through the use of the cutting plane algorithm, particularly in the convergence of the objective to its optimal value. It appears that formulation 1 is the most efficient formulation for collaborative optimization, whereas the efficiencies of formulations 2 and 3 are roughly the same. Formulation 1 is the only formulation employing the l_2 metric. It might be noted that when the metric in formulation 1 was modified to be the sum of the squares rather than the square root of the sum of the squares, its efficiency worsened by a factor of 2 when the SQP algorithm was used and by a factor of 10 when the cutting plane algorithm was used. When the concurrent subspace selfish linear approach did not fail ($A_{coup} = \frac{1}{2}$), the number of evaluations to convergence was comparable to that of the single-level multidiscipline feasible alternating linear approach.

Since the evaluation counter was incremented in the evaluation subroutine for discipline 1 only, the efficiency results would not change significantly if parallelism of the disciplines were exploited. However, the efficiency results in terms of the evaluation counter are not telling the whole story. In the study, the actual execution time to convergence for the single-level all-at-once approach was about as long as that of the collaborative optimization approaches and more than an order of magnitude longer than that of the single-level multidiscipline feasible approaches. The reason for this is that model evaluation for the synthetic test problems is trivial, and most of the execution time was spent in the numerical procedures of the optimizers. Since the single-level all-at-once approach solved the biggest optimization problem, it required a lot of execution time. It is anticipated that in most real-world applications model evaluation will be significant rather than trivial.

Acknowledgments

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